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CONNECTED COMPONENTS OF MODULI SPACES

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0. Introduction

Let S be a minimal surface of general type (complete and smooth over $\mathbb{C}$), and let $\mathscr{M} = \mathscr{M}(S)$ (resp., $\mathscr{M}^{\text{diff}}$) be the coarse moduli space of complex structures on the oriented topological (resp., differential) 4-manifold underlying S.

By Gieseker's theorem [5], $\mathscr{M}(S)$ is a quasiprojective variety, and the number $\nu(S)$ of its irreducible components is bounded by a function $\nu_0(K^2, \chi)$ of the two (topological) invariants $K^2 = K_S^2$, $\chi = \chi(\mathscr{O}_S)$.

Let $\lambda(S)$ be the number of connected components of $\mathscr{M}(S)$: this short note answers a question raised in a previous paper [1], showing that the above number $\lambda(S)$ can be arbitrarily large.

As in [1], to which we shall constantly refer, again we restrict our attention to bidouble (i.e., Galois with group $(\mathbb{Z}/2)^2$) covers of $Q = \mathbb{P}^1 \times \mathbb{P}^1$: indeed, (cf. [2]) we conjecture a stronger result to hold true, namely that many of the different irreducible components of $\mathscr{M}$ we thus obtain are in fact connected components of $\mathscr{M}$.

The idea of proof is rather simple: if S and S' are deformations of each other, then there exists a diffeomorphism $f\colon S \to S'$ such that $f^*(K_{S'}) = K_S \in H^2(S, \mathbb{Z})$, and, in particular, if $r(S) = \max\{r \in \mathbb{N} \mid (1/r)K_S \in H^2(S, \mathbb{Z})\}$, then $r(S) = r(S')$.

In view of Donaldson's recent result [3], it is possible that the integer $r(S)$ could be an invariant of the differentiable structure for these surfaces; it is not clear at the moment whether nicer properties are enjoyed by the moduli spaces $\mathscr{M}^{\text{diff}}(S)$. Nevertheless, when the complex dimension is at least 3, it seems (cf. [6], [7]) that similar phenomena of high disconnectedness should appear also for $\mathscr{M}^{\text{diff}}$.

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1. Statement and proof of the main result

Theorem. *For each natural number k there exist minimal models $S_1, \cdots, S_k$ of surfaces of general type such that*

(a) *S_i is simply-connected ($i = 1, \cdots, k$),*

(b) *for $i \neq j$, S_i and S_j are (orientedly) homeomorphic but not a deformation of each other.*

Remark 1. From the proof it shall also follow that the number of moduli of S_i (cf. [1, p. 484]) differs from the number of moduli of S_j for $i \neq j$.

Let us recall an arithmetical result, proved by E. Bombieri in the Appendix to [1].

Lemma 2. *For each positive integer k, there exist integers m, T, and k distinct factorizations of 6^m,*

$$u_i' v_i' = 6^m \qquad (i = 1, \cdots, k),$$

together with integers w_i, z_i ($i = 1, \cdots, k$) such that, setting $u_i = Tu_i'$ and $v_i = Tv_i'$, the following system of equalities and inequalities is satisfied for $i = 1, \cdots, k$:

$$u_i v_i = T6^m = M, \quad w_i z_i - 2(u_i + v_i) = N,$$
$$(u_i + 2)/3 < w_i < u_i - 4, \quad (v_i + 2)/3 < z_i < v_i - 4.$$

Corollary 3. *In the notations of Lemma 2, the greatest common divisors (u_i, v_i) assume at least $k/2$ distinct values.*

Proof. Set $u_i' = 2^{x_i} 3^{y_i}$. We can clearly assume $x_i \leqslant m - x_i$, hence $(u_i, v_i) = T2^{x_i} 3^{\min(y_i, m - y_i)}$. Since the factorizations are distinct, $(u_i, v_i) = (u_j, v_j)$ for $i \neq j$ if and only if $x_i = x_j$, $y_i = m - y_j$. q.e.d.

Given a smooth projective variety X we denote by $\mathrm{NS}(X)$ the Neron-Severi group of divisors modulo numerical equivalence (which we shall denote by $\sim$, leaving the symbol $\equiv$ for linear equivalence). Note that, more generally on a compact complex manifold X,

$$\mathrm{NS}(X) = \left(\ker\left(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)\right)\right)/\text{torsion}.$$

Lemma 4. *Let X, Y be smooth projective varieties and let $\pi : X \to Y$ be a finite Galois cover with group G. Then*

(i) *$\pi^* : \mathrm{NS}(Y) \to \mathrm{NS}(X)$ is injective, maps to $L = (\ker \pi_*)^\perp = \mathrm{NS}(X)^G$, and $L/\mathrm{im}\,\pi^*$ is a torsion subgroup of exponent at most the order of G.*

If $B_1, \cdots, B_k$ are the irreducible components of the branch divisor B of π, let (for $i = 1, \cdots, k$) e_i be the order of the inertia group of any divisor in $\pi^{-1}(B_i)$, let d_i be the order of divisibility of the class of B_i in $\mathrm{NS}(Y)$ ($d_i = \max\{d \mid \exists \Gamma_i \text{ s.t. } d\Gamma_i \sim B_i\}$), and set $m_i = \text{g.c.d.}(e_i, d_i)$, $a_i = e_i/m_i$.

Assume furthermore $H_1(X,\mathbb{Z})=0$ *and* $H^2(G,\mathbb{C}^*)=0$ (*e.g., if G is cyclic*). *Then*

(ii) *the exponent* β *of* $L/\mathrm{Im}\,\pi^*$ *is the least common multiple* α *of the numbers* $a_1,\cdots,a_k$.

Proof. If m is the order of the group G, we have $\pi_*\pi^*=m$ (Identity), hence π^* is injective, and $\mathrm{im}\,\pi^*\subset(\ker\pi_*)^{\perp}$ by the projection formula $\pi^*x\cdot y=x\cdot\pi_*y$. Moreover, since $\pi^*\pi_*=\sum_{g\in G}g^*$, tensoring over $\mathbb{Q}$, $\ker\pi_*$ is the kernel of the projector onto the subspace of invariants, and $(\ker\pi_*)^{\perp}=\mathrm{NS}(X)^G$. If $x\in L$, then $g^*x=x\ \forall g\in G$; hence $mx=\pi^*(\pi_*x)$ and the first assertion is proven.

Since $H_1(X,\mathbb{Z})=0$, any element x in L is represented by a divisor D s.t. $g^*D\equiv D\ \forall g\in G$.

Consider the sheaf $\mathscr{L}=\mathcal{O}_X(D)$ of rational functions f with $\mathrm{div}(f)-D\geqslant 0$: by assumption, $\forall g\in G$ there exists an isomorphism between $\mathscr{L}$ and $g^*\mathscr{L}$, hence, defining $G(\mathscr{L})=\{(g,\tilde{g})\,|\,g\in G$ and $\tilde{g}$ is an isomorphism from $g^*\mathscr{L}$ to $\mathscr{L}\}$, we have a central extension

$$0\to\mathbb{C}^*\to G(\mathscr{L})\to G\to 0. \tag{5}$$

We notice that

Sublemma 6. (5) *splits if and only if D is linearly equivalent to a G-invariant divisor* D' (*i.e.,* $g(D')=D'\ \forall g\in G$).

Proof. The "if" part is obvious, since then $\mathscr{L}\cong\mathcal{O}(D')$ and the condition $\mathrm{div}(f)-D'\geqslant 0$ is clearly G-invariant, hence there is an action of G on $\mathscr{L}$ which makes (5) split. Conversely, if (5) splits there is an action of G on $\mathscr{L}$, and the sheaf $(\pi_*\mathscr{L})^G$ is nonzero.

If H is a very ample divisor on $Y=X/G$, for $m\gg 0$ the sheaf $(\pi_*\mathscr{L})^G(mH)$ has a section, hence $D+m\pi^*H$ is linearly equivalent to an effective divisor C, which is G-invariant. q.e.d.

Now the extensions of G by $\mathbb{C}^*$ are classified by $H^2(G,\mathbb{C}^*)$; hence, if $H^2(G,\mathbb{C}^*)=0$, D is linearly equivalent to a G-invariant divisor, and we can only consider the case of an effective G-invariant divisor C. In this case, if $R_i=\pi^{-1}(B_i)_{\mathrm{red}}$, we can write C as $C=C_R+C'$, where C_R, C' are effective, no component of the ramification divisor R appears in C', and $C_R=\sum_{i=1}^k b_iR_i$ (since $g(C)=C\ \forall g\in G$, this is possible).

We have

$$\pi_*(C)=m\Gamma'+\sum_i(b_im/e_i)B_i\sim m\Gamma'+\sum_i(b_id_im/e_i)\Gamma_i$$

since B_i is exactly d_i-divisible.

Now

$$mC\equiv\pi^*\pi_*(C)\equiv m\pi^*(\Gamma')+\sum_i(b_id_im/e_i)\pi^*(\Gamma_i),$$

hence

$$\alpha C \equiv \pi^*(\alpha\Gamma') + \sum_i (\alpha/a_i)b_i(d_i/m_i)\pi^*(\Gamma_i),$$

thus αC belongs to $\operatorname{im}\pi^*$.

Conversely, we claim that the class of R_i in $L/\operatorname{im}\pi^*$ has period exactly equal to a_i.

In fact $a_i R_i \equiv (d_i/m_i)\pi^*(\Gamma_i)$, as we have seen, and if there exists a divisor Γ and some integer c dividing a_i such that $cR_i \equiv \pi^*(\Gamma)$, applying π_* we get

$$m\Gamma \sim c\pi_*(R_i) \sim (cm/e_i)B_i.$$

Hence $e_i\Gamma \sim cB_i \sim cd_i\Gamma_i$, thus $(e_i = a_i m_i!)$ $a_i m_i \Gamma \sim (cm_i d_i/m_i)\Gamma$ and $(d_i/m_i)\Gamma_i \sim a_i/c\Gamma$. Since d_i/m_i and a_i/c are relatively prime, Γ_i is a_i/c divisible, therefore $a_i = c$.

Remark. The above proof shows that, in general, $\beta \geqslant \alpha$.

Corollary 7. *Let $\pi: X \to Y$ be a finite Galois cover with group G s.t. π is the composition of Galois covers as in* (ii) *of Lemma* 4, *each such that the corresponding integer α equals* 1. *Then* $\mathrm{NS}(X)^G = \pi^*(\mathrm{NS}(Y))$.

Proof. The proof is by induction on the number of steps: in fact if N is a normal subgroup of G and $Z = X/N$ is smooth, let $p: X \to Z$, $q: Z \to Y$ be the quotient morphisms. Since p^* and q^* are injective by Lemma 4, we can identify $\mathrm{NS}(Y)$ and $\mathrm{NS}(Z)$ to subgroups of the free abelian group $\mathrm{NS}(X)$.

Let $\Gamma = G/N$: by induction $\mathrm{NS}(Y) = \mathrm{NS}(Z)^\Gamma = (\mathrm{NS}(X)^N)^\Gamma = \mathrm{NS}(X)^G$. q.e.d.

Remark. The result of Corollary 7 can be stated in a greater generality, in particular one does not need the intermediate quotients to be smooth.

Proof of the theorem. Recall that $\pi: S \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth simple bidouble cover of type $(2a, 2b)$, $(2n, 2m)$ if π is a finite $(\mathbb{Z}/2)^2$ Galois cover, S is a smooth surface, and the branch locus of π consists of two curves of respective bidegrees $(2a, 2b)$, $(2n, 2m)$.

Apply Lemma 2 to the integer $2k$. Then, for $i = 1, \cdots, 2k$ set (in the notations of the lemma)

$$a_i = (u_i + w_i)/2 + 1, \quad n_i = (u_i - w_i)/2 + 1,$$
$$b_i = (v_i - z_i)/2 + 1, \quad m_i = (v_i + z_i)/2 + 1,$$

and let, for $i = 1, \cdots, 2k$, $\pi_i: S_i \to Q$ be a smooth simple bidouble cover of type $(2a_i, 2b_i)$, $(2n_i, 2m_i)$. As in [1], p. 506] we see that

$$K_{S_i}^2 = 8M, \qquad \chi(\mathcal{O}_{S_i}) = \tfrac{3}{2}u_i v_i + (u_i + v_i) + 2 - \tfrac{1}{2}w_i z_i = \tfrac{3}{2}M + 2 - \tfrac{1}{2}N.$$

Moreover,

$$K_{S_i} = \pi_i^*(\mathcal{O}_Q(u_i, v_i)) \tag{8}$$

and, u_i, v_i being even, K_{S_i} is 2-divisible: hence, by Freedman's theorem [4] (cf. also [1, Theorem 4.6]), all the surfaces S_i are (orientedly) homeomorphic. Applying Corollary 6 to $\pi_i: S_i \to Q$, and using (8), we see that $r(S_i) = \max\{r \in \mathbb{N} \mid (1/r)K_{S_i} \in H^2(S_i, \mathbb{Z})\}$ equals the greatest common divisor (u_i, v_i).

By Corollary 3 there are at least k of the $2k$ surfaces $S_1, \cdots, S_{2k}$, which satisfy the requirements of the theorem (S_i is simply connected by [1, Proposition 2.7]), since $r(S)$ is a deformation invariant, as is easy to show.

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